Mathematical Foundations of Game Theory

By: Michael Bianco

Submitted To:

The Mathematics & Economics Departments of Franciscan University

Table of Contents:

Abstract:

The recently developed field of Game Theory has occupied the minds of many contemporary mathematicians and economists alike. This paper will explore the mathematical foundations of Game Theory, solutions to common strategic situations, as well as provide some application to specific economic situations.

I. Introduction

A. Basic Definition

In order to assist our inquiry into the mathematical foundations and economic application of game theory a definition of terms is necessary. Despite the fact the fact that the word 'game' exists in the title of the field of study, the application of game theory is not restricted exclusively to what we would think of as games (i.e. chess, checkers, etc).

According to *Library of Economics and Liberty* the object of game theory is to "determine mathematically and logically the actions that 'players' should take to secure the best outcomes for themselves in a wide array of 'games.'" (Dixit & Nalebuff, 2008). Games are interactions between two or more *players* that are characterized by situations in which a player's outcome at the of the game is dependent on the decisions of the other players. An individual decision or choice of a player is defined as a *move*, a series of moves of a given player is a *strategy*. A unique combination of players' strategies will result in a game *outcome.* The result of a game for a given player is defined as the *payoff* (Straffin, 1996, P.3). A *player* is defined as a rational agent (not necessarily a person, could be a institution or firm) where rationality consists of "complete knowledge of one's interests, and flawless calculation of what actions will best serve those interests" (Dixit, Reiley, and Skeath, 2009, P. 30). With knowledge of possible game outcomes and payoffs associated with those outcomes, it is often possible to find a *equilibrium* (more specifically a *nash equilibrium*) for that game. There are many types of equilibrium but in a broad sense it can be defined as the state when "each player is using the strategy that is the best response to the strategies of the other players" (Dixit, Reiley, and Skeath, 2009, P. 33).

B. Historical Development

Antoine-Augustine Cournot

Although many would claim that the development of game theory is rooted in the 20th century work of John von Neumann, others would argue that the father of game theory is Antoine-Augustine Cournot (Morrison, 1998). Cournot initially began his career as a mathematician, publishing many papers that caught the attention of fellow mathematician Poisson, and eventually secured professorship at the University of Lyons. His mathematical background uniquely positioned him to approach the study of Economics with a symbolic and graphical representation of thought, enabling him to present clear definitions of ideas that previously were imprecisely elucidated using the literary form of economic explanation. Although Cournot was not the first to bring mathematics to the study of economics, he is the first to popularize and defend its use.

Cournot "originated an early form of game theory" (Ekelund & Hebert , 2007, P. 571) in his 1838 publication of *Researches into the Mathematical Principles of the Theory of Wealth* through his discussion of competition between mineral water sellers (Ekelund & Hebert, 2007, P.268). He developed the reaction curve, a tool used to predict the output of a firm based on given knowledge of a competing firm's output level. By partially differentiating a firm's output function by the quantity produced of the competing firm and setting the resulting equation equal to zero, he was able to determine the stable equilibrium (outcome) of competing players (firms) in a competitive situation

(Morrison, 1998). This inquiry into interactions between competing agents was the first of much analysis undertaken to understand the nature of competitive environments.

Neumann, Nash, and Others

Although there were a couple thinkers (such as Joseph Bertrand in 1883 and Francis Ysidro Edgeworth in 1897) who continued to build upon the foundation which Cournot created, substantial strides in the development of game theory were not made until John von Neumann starting publishing his work. A brilliant man, Neumann earned a doctorate in chemistry *and* mathematics *at the same time* before the age of twenty-four. He took a professorship position for a short time at the University of Berlin before becoming the youngest member of the Institute for Advanced Study at Princeton (Fonseca). In 1937 Neumann, with the encouragement and help of Oskar Morgenstern, published *Theory of Games and Economic Behavior* which was the first work to assert that "*any* economic situation could be defined as the outcome of a game between two or more players" ("John von Neumann", 2008). In addition to making critical contributions to game theory (such as the minimax theorem), Neumann also assisted in the development of the first computer and was a key member of the Manhattan Project (Myhrvold, 1999).

John Forbes Nash Jr., popularized by the recent movie *A Beautiful Mind,* was another pivotal contributor to the field of game theory in the twentieth century. From 1950 – 1953 he published a series of four papers, which eventually won him the Nobel Prize in 1994, in which he proved that in non-cooperative (i.e. competitive) games "at least one equilibrium exists as long as mixed strategies are allowed" ("John F. Nash Jr", 2008). This specific type of equilibrium that can be found in non-cooperative games was later coined the *nash equilibrium.* In addition to his work on non-cooperative games, he also made great strides in the area of cooperative games (i.e. bargaining or auction situations). Nash's work, along with those who built upon his work, was used in the fairly recent electromagnetic spectrum auction: "a multiple-round procedure was carefully designed by experts in the game theory of auctions…The result was highly successful, bringing more than $10 billion to the government while guaranteeing an efficient allocation of resources" (Milnor, 1998). Correctly applied game theoretic principals to real-world situations can yield great results.

II. Game Setup

A. Strategies & Payoffs

Although the definition of *strategy* worked well for an initial investigation, the idea will need to be further nuanced by introducing the terms *pure strategy* and *mixed strategy*. A given player *A* has *m* strategies where $\{ \pi_{Ai} \mid i = 0,1,\ldots,m \}$ represents the set of all strategies available for player A. In addition to choosing a single strategy – a *pure strategy* – a player can choose a *mixed strategy*. A mixed strategy is represented by an *m*-tuple $u = (x_1,x_2,\ldots,x_m)$ of non-negative real numbers such that $\sum_{i=1}^{m} x_i = 1$ where x_i represents the probability that strategy π_{Ai} will be played (sometimes this *m*-tuple is referenced as a *strategy profile*). In other words, a mixed strategy is a probability distribution over the set of all possible strategies. With the exception of the trivial case where $m = 1$, there exists infinitely many mixed strategies. With a precise definition of mixed strategies in place, it can be easily seen that a pure strategy is simply a special case of a mixed strategy where there exists $x_j = 1$ such that $x_i = 0$ where $i \neq j$.

These distinctions between strategy types might seem a bit abstract, but some investigation will reveal their operation in real world situations. Encouraging students to learn can be seen as a game between teachers and students. Teachers want students to come to class, but they would rather not spend time taking roll and calculating an attendance grade at the end of the year. In order to minimize time spent on punishing students for lack of attendance and maximize the number of students in the class, teachers will often check attendance sporadically – utilizing a mixed strategy – in order to achieve their desired result. If a teacher used a pure strategy – checking attendance using some fixed pattern – students would detect the pattern and simply skip class when teacher did not check attendance, ultimately lowering attendance and creating more work for the teacher.

The *payoff* resulting from a chosen pure strategy is dependent on the chosen strategies of the other players and is measured in a real number value representing the utility for the given player resulting from the game's outcome (e.g. 'usefulness', not necessarily a monetary measure). When playing a mixed strategy it is impossible to calculate the exact payoff given other players chosen strategies, however utilizing the expected value concept from probability we can calculate the *expected payoff* of a given mixed strategy.

Let $p(u_A, s_B)$ represent the expected payoff for player A, assuming mixed strategy probability distribution $u = (x_1, x_2, \ldots, x_m)$ for A and $s = (y_1, y_2, \ldots, y_n)$ for B. Let c_{ij} represent the payoff for player A associated with player A choosing pure strategy and π_{Bi} and player B choosing pure strategy π_{Bj}, where A has m number of pure strategies and B has n. Recall from probability theory that the expected value of a random variable X can be represented as $E(X) = \sum_{i=1}^{n} P(X = x_i)x_i$ assuming the number of trials approaches infinity (Mendelson, 2004, P.214). Let $X = \{ c_{ij} \mid i = 0 \ldots m, j = 0 \ldots n \}$ and $P(X = x_i) = P(X = c_{ij}) = x_i y_j$, given u and s the expected payoff would be $E(X) = \sum_{c_{ij} \in X} c_{ij} x_i y_j$. A general definition for the expected payoff can now be provided from the perspective of player A:

$$p(u,s) = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_i y_j .$$

To notation above is a bit restricted because it is examining the game from the perspective of player A. To differentiate the payoff function between A and B, $p_A(u,s)$ will be used to represent the payoff function for A and $p_B(u,s)$ for B. When dealing with the payoff function, all mixed strategy distributions for all other players will be given or previously defined. With this in mind, notational simplifications can be made to the payoff function, defining the payoff function for a player i as $p_i(u)$. The set of all possible mixed strategies (remember that a pure strategy is just a special case of a mixed strategy) for a player i will be defined as S_i.

B. Graphical Display

Extensive Form

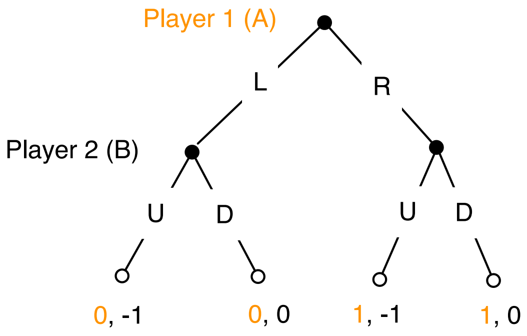

Figure 1

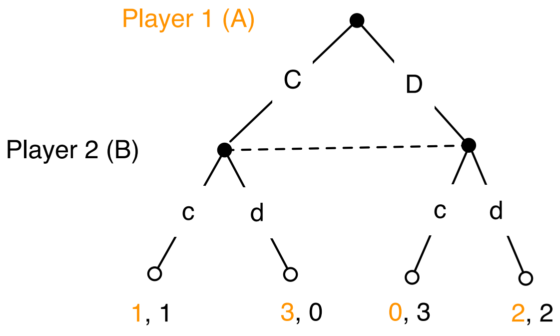

Figure 2

The extensive form of a game, also known as a game or decision tree, is a graphical representation of the decisions and sequence of those decisions that constitute players' strategies in a game. Game trees give a 'big picture' overview of the progression that a game takes; it provides a timeline in addition to a set of possible strategies a player can take in one easily interpreted display. The games above display the decisions of two players, but the extensive form of a game can easily accompany more than two players.

The extensive form of a game is made up of decision points, *nodes* (or *vertexes* in graph theory terminology), at which the player must chose between one or more moves represented by *branches*. In figure 1 depicted above, player 1 (who will be referred to as *A* from now on) must chose between choosing left or right and player 2 (*B*) – with full knowledge of *A*'s choice – chooses between moves U and D. In both of the above diagrams, the end of the game is indicated by the 'open' node – the *terminal node* – at which a payoff for each player is given. Figure 2 differs from figure 1 in that *A* makes a decision without knowledge – with *imperfect information* – of what *B* has chosen; *A* and *B* are making *simultaneous moves* (as opposed *sequential moves*) indicated by the dotted line connecting the nodes. The moves that *B* must choose from in figure 2 are said to be in the same *information set*. If all information sets contain only one element (as in figure 1) then the game is said to have *perfect information* (Kuhn, 1953).

Making the distinction between games with perfect and imperfect information is crucial to determining if a equilibrium exists. For instance, a game with perfect information "always has an equilibrium point in pure strategies" (Kuhn, 1953, P. 209). Understanding what information is available can enable you determine the outcome of a game with a much higher level of accuracy.

The level of information available to a player completely changes the decision making process – information is one of the most valuable assets one can possess. One of the most interesting games playing itself out in the market is the 'phone war' between Google, Apple, RIM, and others. Few firms have the know-how, leadership, capital, and monopoly power required in order to create a competitively priced smart phone with accompanying software and adequate feature set. Additionally, the labor pool with adequate education and experience in designing both the software and hardware end of a smartphone is limited. This unique situation creates an environment where the price on intellectual property, talented employees, and product secrecy soars. Employees working on new projects at Apple are often video monitored and "must pass through a maze of security doors, swiping their badges again and again and finally entering a numeric code to reach their offices" (Stone & Vance, 2009). Apple has a dedicated "Worldwide Loyalty Team" to ensure product secrecy and eliminate information leaks. Misinformation about products is intentionally spread internally and bloggers who leak product information are quickly sued.

In order to attract talented employees and keep hired talent, Google recently gave "all its employees a $1,000 holiday bonus in addition to pay increases of at least 10 percent" (Comlay, 2010). The amount of capital invested in the attainment and protection of information in oligopoly markets is enormous, game theory gives us key insights into what information is valuable and helps determine how to properly value that information.

Normal Form

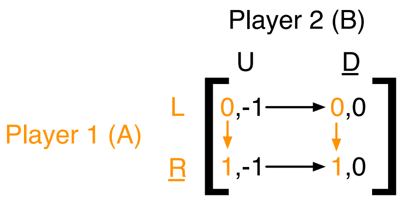

Figure 3

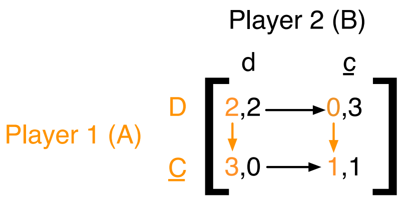

Figure 4

The normal form of a game, also known as the strategic form or game matrix, is characterized by a matrix-like display of payoffs available to players. This matrix-like display of payoffs is not simply a coincidence, as we will discuss later, linear algebra is used to find solutions (equilibrium) of games. The normal form of a game *cannot* depict time related information – all moves displayed are simultaneously made. With this in mind, it is important to note that although the payoffs of the game in figure 3 mirror the payoffs of the game in figure 1 they *do not* represent the same game. However, the game in figure 4 *does* represent the same game as figure 2; the information set contained in figure 2 allows it to be properly represented as a simultaneous game.

C. Game Characteristics

Constant and Non-Constant Games

How payoffs change when other players' strategies change determines whether we classify a game as *constant* or *non-constant*. All the games depicted thus far have been non-constant sum games: a increase in the payoff of one player did not necessarily require a decrease in the other player's payoff. In constant-sum games there is some fixed net payoff amount, increase in one player's payoff necessitates a payoff decrease for other players'. A special case of the constant-sum game is the *zero-sum* game, where the sum of the players' payoffs is zero. The terms constant & zero-sum are often used interchangeably since a constant-sum game can be translated to a zero-sum game by simply adjusting the utility scale. Zero-sum games are commonly found in what one would traditionally consider a game: chess, checkers, soccer, basketball, etc. In games such as chess or basketball there cannot be two winners: one team wins the other loses.

Zero-sum games have a very special implication: they are *strictly competitive*. Because there can only be one winner, players have no incentive to cooperate. Characterizing a game as zero-sum has both effects on mathematical analysis as well as a players' mindset. Mathematically, classifying a game as zero-sum allows certain analysis (such as *minimax* analysis, which will be discussed later) to be performed and enabled certain judgments to be made about the equilibrium. Understanding how you – and your competitors – are classifying a situation will change the strategies players' will chose and change the way you should predict other players' decisions. For instance, many people falsely characterize the interaction of entrepreneurs in the free market economy as a zero-sum game. This changes how a given entrepreneur will view and interact with others;

instead of viewing them as potential future business partners they are categorized as competition to be beaten. The free market is not a zero-sum game, although there are some businesses that will directly compete with each other, for the most part businesses can interact and collaborate in order to increase wealth rather than ‘steal’ it from each other.

Cooperative & Repeated Games

Although we will not explore these concepts in detail it is useful to give a brief overview of these two important concepts. A cooperative game is one in which communication between players is allowed, they are “permitted to communicate prior to playing the game, to make binding agreements” (Luce & Raffia, 1957, P. 152) in addition to many other communication tactics. Cooperative games are characteristic of many political situations and contract negotiations between businesses.

The number of times a game is played can greatly effect the strategies which players will choose to employ. When a player repeatedly plays a board game against the same opponent the player will start to notice certain patterns in his opponent’s thought and utilize strategies that will work well against his opponent. The notion of *finitely* repeated and *infinitely* repeated games are crucial to comprehensive game theoretic analysis. All games presented here will be assumed to be one-shot games (played only once).

D. Characterizing Solutions

Nash Equilibrium

A general and broad definition of equilibrium was given earlier, the concept of a *nash equilibrium* will now be fleshed out in more detail. Remember that a strategy that is a nash equilibrium is not always equivalent to the strategy choice that will result in the highest possible payoff for players'. However, it is the best possible response to the expected choices of all other players involved in the game. In other words, a nash equilibrium exists if, for every player, a change in that player's strategy will not improve his payoff holding all other players' strategy choices constant.

The idea of a nash equilibrium can be furthered by introducing a *mixed strategy nash equilibrium* and *pure strategy nash equilibrium*. These distinctions highlight the fact that a nash equilibrium is not always a pure strategy, but can be an optimal mix of many pure strategies engineered to generate a maximum expected payoff. Although every every game has a one or more mixed strategy nash equilibrium, it is possible for a game not to have any pure strategy nash equilibrium. Often a nash equilibrium strategy will be called a equilibrium point or (for two player games) an equilibrium pair.

Mathematical methods of finding the nash equilibria of a game will be investigated later, but for now it is helpful to see the logic of the nash equilibrium operating in figure 3 and figure 4. The arrows represent *A & B*'s 'thinking', they graphically describe the logic used in making the best decision, each player will want to chose the option that will provide them with the highest payoff. Notationally, the nash equilibrium of *A* choosing strategy *R* and *B* choosing *D* (in figure 3) will be denoted as *(D, R)*.

Looking closer at the chosen strategies, it becomes apparent that in figure 4 both players receive the best result possible, but in figure 4 the chosen strategies (c, C) result in a less than optimal outcome (the best possible outcome would have been (d, D)). This type of game is called the prisoner's dilemma. Both players (prisoners) chose what is best for them (confess) and both end up receiving a lower payoff.

Aside from the jailhouse, we can see the prisoner's dilemma operating in oligopoly markets. In a market with few sellers, some amount of monopoly power exists allowing the firm to charge more for their product than they would if there were many sellers. In other words, the firm's marginal revenue exceeds its marginal cost for each product or service sold. However, competitors in a oligopoly have a natural incentive to decrease prices in order steal customers from the other firm, ultimately benefiting consumers. The perceptive firm realizes that if they could come to a price fixing agreement then both firms would benefit in the long run (and consumers would lose). This is why, in order to protect consumers, there are anti-trust laws in place – to decrease the potential gains of collusion by associating penalties to the strategy of price-fixing (Hawkins, 2009).

Thus far a colloquial definition of the nash equilibrium has been given, a precise mathematically definition will now be given. Let q be the number of players in a game, i be an integer such that $0<i \leq q$, S_i be the set of available mixed strategies for player i, $u_i \in S_i$. Let $(u_1,\ldots,u_q)$ be an q-tuple corresponding to another q-tuple $(c_1,\ldots,c_q)$ where $c_i=p_i(u_i)$. A q-tuple $(u_1,\ldots,u_q)$ (which represents all players' chosen strategies) is a nash equilibrium if $\forall i, \forall u_i^* \in S_i, p_i(u_i^*) \geq p_i(u_i)$.

Dominance

A strategy is said to *dominate* another strategy if, regardless of what the other players' chose, that strategy will result in a better payoff. A dominant strategy can either *strictly dominate* or *weakly dominate* one or more strategies. A strategy u_i^* strictly dominates strategy u_i if $\forall\ u_1 \in S_1, u_2 \in S_2, \ldots, u_{i-1} \in S_{i-1}, u_{i+1} \in S_{i+1}, u_n \in S_n\ \ p_i(u_i^*) > p_i(u_i)$. A strategy u_i^* *weakly* dominates strategy u_i if $\forall\ u_1 \in S_1, u_2 \in S_2, \ldots, u_{i-1} \in S_{i-1}, u_{i+1} \in S_{i+1}, u_n \in S_n\ \ p_i(u_i^*) \geq p_i(u_i)$. Note that eliminating weakly dominated strategies can also eliminate some nash equilibria.

Determining which strategies are dominated is often essential to being able to solve for the nash equilibrium. Once it is determined that a strategy is dominated then it can be removed from the game matrix, essentially redefining the game, possibly creating opportunities to find a nash equilibrium that could not be previously discovered.

Pareto Optimality

A pareto optimal, or pareto efficient, equilibrium of a game is defined as a solution in which it is not possible to increase the payoff of any player without decreasing the payoff of one or more players. Let q be the number of players in a game, S_i be the set of available mixed strategies for player i, $u_i \in S_i$, $(u_1,\ldots,u_q)$ be an q-tuple corresponding to another q-tuple $(c_1,\ldots,c_q)$ where $c_i = p_i(u_i)$. A equilibrium $(u_1,\ldots,u_q)$ with corresponding payoff q-tuple $(c_1,\ldots,c_q)$ is pareto optimal if $\nexists\ (u_1^*,\ldots,u_q^*)$ with corresponding q-tuple $(c_1^*,\ldots,c_q^*) : \forall\ \ i=1,\ldots,q\ \ c_i^* \geq c_i$ and $\exists$ an integer $0 < j \leq q : c_j^* > c_j$.

From the discussion of the nash equilibrium, it is apparent that although the nash equilibrium *can* be a pareto optimal solution it is not *necessarily* a pareto optimal solution. For example, take the case of the prisoner's dilemma in figure 4: although (c, C) is the nash equilibrium it is *not* a pareto optimal solution (every other possible pure strategy choice in that game *is* pareto optimal). This underscores the fact that the nash equilibrium, in many cases, is *not* the outcome that both parties should desire. If the nash equilibrium is not pareto optimal this could be a sign of a coordination problem, signaling players involved that cooperation should be engaged in order to eliminate any coordination issues that are preventing one or more players from increasing their payoffs without expense to other players involved.

The nash equilibrium, combined with the notion of pareto efficiency, in the case of the prisoner's dilemma highlights an interesting moral issue: self-interest does not always result in the optimal result for both parties. For the hedonist who is strictly concerned about himself, the "pursuit of self-interest backfires, and prevents us from conforming to collectively advantageous social arrangements" (Luper, 2001). The hedonist can take a well-needed lesson from our above discussion; sacrifice and concern for others might result in a higher level of utility for all.

III. Mathematical Analysis

A. Two Person, 3 x 3, Zero-Sum Game

$$\begin{array}{cc|ccc|c} & & & B & & \\ & & U & F & D & \text{Min:} \\ \hline A & L & 2 & 4 & 13 & 2 \\ & S & 6 & 5 & 10 & \underline{5} \\ & U & 6 & 4 & -1 & 1 \\ \hline & \text{Max:} & 6 & \underline{5} & 13 & \end{array}$$

Figure 5

$$\begin{array}{c|cc|} & U & F \\ \hline L & 2 & 4 \\ S & 6 & 5 \\ \hline \end{array}$$

Figure 6

$$\begin{array}{c|cc|} & G & C \\ \hline E & 3 & 1 \\ F & 2 & 4 \\ \hline \end{array}$$

Figure 7

Solving Using the Minimax Algorithm

Figure 5 represents a two-person zero-sum game that happens to be dominance solvable in addition to solvable in pure strategies. Recall that zero-sum games are strictly competitive: more for one player implies less for other players. Mathematically this can represented by noting the payoff function structure $p_A(u,s) = -p_B(u,s)$. As a notational aid, a *mutual* payoff $M(u,s) = p_A(u,s) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i y_j$ will be used to represent payoffs in zero-sum games. Recall that equilibrium strategies are the best response given the chosen strategies' of other players; a player is essentially choosing the best from all the worst case scenarios. The minimax algorithm is based on this logic: finding the minimum payoff (worst) choices for a player and then choosing the maximum payoff (best from the worst) out of the available choices. The column player in a zero-sum game will always desire the lowest possible payoff since the stated payoff value represents the utility for the row player. Therefore, for the column player the maximum of the minimums (maximin) will be chosen and for the row player the minimum of the maximums.

Examining figure 5 it is easy to see that for *A* the maximin is 5 and for *B* the minimax is 5 resulting in an equilibrium pair *(F, S)*. It is not a coincidence that both values are equal – this is a property of the minimax theorem that will be explored in more detail later. The equilibrium pair that we have found is a pure strategy nash equilibrium. Not every zero-sum game has a pure strategy nash equilibrium, but (as will be shown later) every zero-sum game has a mixed strategy nash equilibrium.

Solving Using Iterated Dominance

Although not always the case, in figure 5 the nash equilibrium can also be found by eliminating dominated strategies and choosing the best remaining strategy. Comparing strategies *S* & *U* of player *B* it is apparent that *S* weakly dominated *U*. Examining *A* finds that both *U* & *F* strictly dominate *D*. The new resulting matrix (depicted in figure 6) makes it clear that *L* is dominated and thus *F* is the best choice for *B* resulting in (F, S) as the equilibrium pair – the same result found using minimax analysis.

Solving Graphically

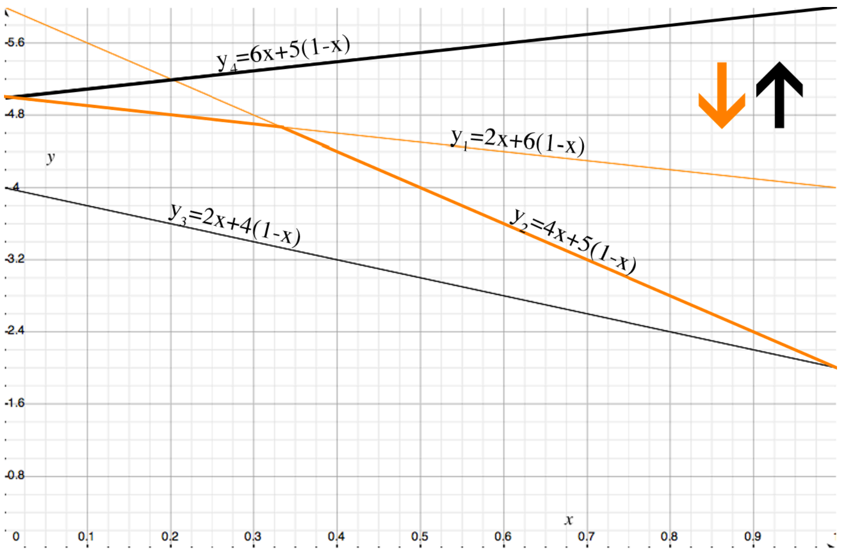

Figure 8

In addition to the other methods of finding an equilibrium point, we can use graphical analysis combined with the logic of the minimax algorithm. A system of linear equations can be derived from the game matrix of any given game. The lines plotted in figure 8 are from figure 7 and represent the expected payoffs for all possible mixed strategies given the strategy choice of another player (this is why there is two lines for each player). Using the logic of the minimax algorithm, for *A* an upper constraint (the bolded 'security line') is created by choosing the minimum value between the two payoff lines at each *x* (which determines the probability distribution between the two strategy choices) along the axis; the maximum between the two payoff lines is chosen for *B*. The intersection of these lines results in the nash equilibrium, in the above figure, consistent with our previous analysis, the minimax value is 5 resulting in an identical equilibrium pair of (S, F).

B. The Minimax Theorem

Definition

The minimax theorem was first released by Neumann in a foreign academic journal *Mathematische Annalen* under the title of *Zur Theorie der Gesellschaftsspiele* which was later translated into english and published in the Princeton journal *Annals of Mathematical Studies* in volume 4 of *Contributions to the Theory of Games*. The theorem below is retrieved from the english translation.

Before diving into the minimax theorem, some additional groundwork must be laid. Earlier, the mutual payoff function was defined as *M* accepting two strategy profiles as arguments. In addition to accepting strategy profiles, *M* can accept a pure strategy argument in place of either strategy profile such that $M(\pi_{A_i},s) = M(\pi_i,s) = \sum_{j=1}^{n} c_{ij}y_j$ (recall that c_{ij} is the payoff from the perspective of *A*, thus *B*'s payoff in a zero-sum game is $-c_{ij}$). With this notational addition in place, the two part minimax theorem can now be stated.

Theorem 1: *For two-person zero-sum games each of the following three conditions implies the other two.*

Condition 1:

An equilibrium pair exists

Condition 2:

$$v_1 = \max_{u \in S_A} \min_{s \in S_B} M(u,s) = \min_{s \in S_B} \max_{u \in S_A} M(u,s) = v_2$$

Condition 3:

There exists a real number v, mixed strategy $u^*=(x_1,\dots,x_m)$ and mixed strategy $s^*=(y_1,\dots,y_m)$ such that:

(a) $\sum_{i=1}^{m} c_{ij}x_i \geq v, j = 1,\dots,n$

(b) $\sum_{j=1}^{n} c_{ij}y_i \leq v, i = 1,\dots,m$

Theorem 2: *For every finite, two-person, zero-sum game, there exists an equilibrium strategy.*

Condition 2 states algorithmically how the minimax & maximin are calculated, also implying that $v = v_1 = v_2$. Condition 3(a) states that there does not exist another strategy available to player *A* that could be chosen to ensure a greater worst-case or 'security' payoff (3(b) ensures the same for player *B*). The proof for theorem one is rather simplistic and follows directly from the previous definitions that have been made, for this reason the proof for part two of the theorem will be the focus.

Proof

The following proof for the minimax theorem is adapted from appendix two of Luce & Raiffa's *Games and Decisions*. Recall the definition of a fixed point for a mapping. A mapping T : X → X has the fixed point property if $\exists\, x \in X : T(x) = x$.

Let $u \in S_A,\ s \in S_B,\ u^* \in S_A,\ s^* \in S_B$

Let T be a transformation that maps mixed strategy pairs into mixed strategy pairs

$$T : S_A \times S_B \rightarrow S_A \times S_B$$

$$T(u,s) = (u',s') = ((x_1,\ldots,x_m'),(y_i,\ldots,y_n'))$$

It will be proved that T has two properties:

(a) u^* and s^* are maximin & minimax strategies *if and only if* $T(u^*,s^*) = (u^*,s^*)$

(b) T has at least one fixed point

T is defined as:

$$x_i' = \frac{x_i + c_i(u,s)}{1 + \sum_{k=1}^{m} c_k(u,s)}$$

$$y_i' = \frac{y_i + d_i(u,s)}{1 + \sum_{k=1}^{n} d_k(u,s)}$$

$$c_i(u,s) = \begin{cases} M(\pi_i,s) - M(u,s) & \text{if } M(\pi_i,s) - M(u,s) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$d_j(u,s) = \begin{cases} M(u,s) - M(u,\pi_j) & \text{if } M(u,s) - M(u,\pi_j) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that c_i , d_i are greater that zero if the pure strategy π_{Ai}, π_{Bi} results in a better payoff (remember that numerically lower is better for *B*) than the mixed strategy *u, s* for players *A* and *B* respectively. Building on c_i and d_i, $x_i \neq x_i'$ and $y_i \neq ys_i'$ if there exists some pure strategy that would result in a better payoff for *A* given *B*'s mixed strategy (or *B* given *A*'s strategy). If $x_i = x_i'$ then we know that *A* is making the best strategy choice possible given *s* (and when $y_i = y_i'$ *B* is making the best strategy choice possible given *u*). A couple conclusions come out of this analysis:

if $T(u^*,s^*) = (u^{*\prime},s^{*\prime}) = (u^*,s^*)$ then (u^*,s^*) is an equilibrium pair

if (u^*,s^*) is an equilibrium pair then

$\forall i,\ c_i(u^*,s^*) = 0$

and $\forall j,\ d_j(u^*,s^*) = 0$

The following proof is 'nested' in the sense that it a multi-part and multi-level proof. To assist in following the logic of the proof the three main sections of the proof will be separated into A, B, C. Part B is separated into I & II.

A. Prove: *u′ & s′* are Probability Distributions

Before going any further it must first be proved that the mapping *T* results in a pair of probability distributions. Recall that for a *m*-tuple $(x_1,x_2,\ldots,x_m)$ to be a probability distribution $\sum_{i=1}^{m} x_i = 1$.

$$\sum_{i=1}^{m} u_i' = \sum_{i=1}^{m} \frac{x_i + c_i(u,s)}{1+\sum_{k=1}^{m} c_k(u,s)} = \frac{\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} c_i(u,s)}{1+\sum_{k=1}^{m} c_k(u,s)}$$

We know $(x_i,...,x_m)$ is a probability distribution $\therefore \sum_{i=1}^{m} x_i = 1$

$$= \frac{1 + \sum_{i=1}^{m} c_i(u,s)}{1 + \sum_{k=1}^{m} c_k(u,s)} = 1$$

B. Prove: Property *a* of *T*:

B. I. Assume: *T(u,s) = (u,s)*

Prove: *u* and *s* are maximin & minimax (i.e. equilibrium pair)

In order to prove that *(u, s)* is a equilibrium pair it must first proved that:

$\exists i : x_i > 0$ and $c_i(u,s) = 0$

This will be proved by first showing (by contradiction) that for some *i,* $M(u,s) \geq M(\pi_i,s)$

An alternate definition of *M* in terms of *M(π*$_i$*,s)*: $M(u,s) = \sum_{i=1}^{m} x_m M(\pi_i,s)$

Assume: $\forall i : x_i > 0, M(u,s) < M(\pi_i,s)$

Prove: $\exists i : M(u,s) \geq M(\pi_i,s)$

$M(\pi_i,s) = \sum_{i=1}^{m} x_i M(\pi_i,s) = M(u,s) \sum_{i=1}^{m} x_i = M(u,s)$ ($\pi_i = u$ when a pure strategy is chosen)

$M(u,s) < \sum_{i=1}^{m} x_m M(\pi_i,s)$ Contradiction! $\therefore\ \exists$i : $M(\pi_i,s) \geq M(u,s)$

if $\exists i : M(\pi_i,s) \geq M(u,s)$ then for that *i*, $M(u,s) - M(\pi_i,s) \leq 0 \Rightarrow c_i(u,s) = 0$

Given that *T* has a fixed point $\exists i : x_i > 0,\ x_i' = x_i \therefore x_i' = x_i = \dfrac{x_i + 0}{1 + \sum_{k=1}^{m} c_k(u,s)}$

$\therefore \sum_{k=1}^{m} c_k(u,s) = 0$. By definition of c it is known that $\forall i,\ c_i(u,s) \geq 0 \therefore \forall k, c_k(u,s) = 0$. Again, by the definition of c, this implies $\forall \pi_i,\ M(u,s) \geq M(\pi_i,s)$. This statement is made independent of $s \therefore \forall u^* \in S_A,\ s \in S_B,\ M(u,s) \geq M(u^*,s)$. This same proof can be done for s independent of u. Thus it has been proved that when *(u,s)* is a fixed point for mapping T then *(u,s)* is an optimal strategy (equilibrium pair).

B. II. Assume: *u & s* are an equilibrium pair

Prove: *T(u,s) = (u,s)*

Recall the conclusions made earlier, assuming an equilibrium pair it is known:

$\forall i,\ c_i(u^*,s^*) = 0$ and $\forall j,\ d_j(u^*,s^*) = 0$

In part I of this proof it was shown that when the conditions stated above hold:

$\forall i,\ x_i' = x_i$ and $\forall j,\ y_j' = y_j \ \therefore\ T(u,s) = (u,s)$

C. Prove: property *b* of *T*:

The Brouwer Fixed Point Theorem: *If a function maps each point of a sphere S (interior plus boundary) located in a Euclidean space of finite dimension into another (not necessarily distinct) point of S and if the function is continuous, then there exists at least one point which is mapped into itself.* (Luce & Raffia, 1957, P.392).

It is outside the scope of this paper to do the topological analysis necessary to both prove the Brouwer fixed point theorem[1] and ‘strictly’ prove that it applies to our specific situation. However, sufficient evidence will be provided in order to *almost* prove that the Brouwer fixed point theorem applies. Although the space in which T is operating is not a sphere in a strict sense, assuming we are examining a sphere in a 2 dimensional

---

[1] For a full and detailed proof of the Brouwer fixed point theorem refer to *Dimension Theory* by Witold Hurewicz and Henry Wallman

place, if we "don the topologist's glasses it can be made to look like one" (Luce & Raffia, 1957, P.392)[2]. From earlier analysis, it is self evident that the transformation is continuous, and by the definition of *T* we know that the source and target space are identical. From part A of the minimax theorem proof we know that $\forall u \in S_A,\ s \in S_B,\ T(u,s) = (u',s') \in S_A \times S_B$. Building on the work and proofs completed by others, this is sufficient to show that *T* does indeed always contain at least one fixed point in a finite, zero-sum, 2-person game.

C. Two Person Zero-Sum 2 x 2 Game With Mixed Strategies

Solving Using Systems of Linear Equations

Although we hinted to this earlier when solving figure 7 using the graphical method, solving for a nash equilibrium using systems of linear equations will now be examined in a bit more detail. Minimization and maximization problems, which are at the core of the minimax theorem, are essentially linear programming problems. The analysis below will be limited to a 2-person 2x2 game, but the concepts established can be extended to operated an a 2-person *m x n* game, and with an advanced understanding of linear algebra can be extended to *n*-person games with finite pure strategies.[3]

For a 2-person 2x2 game, there are two separate linear algebra problems that must be solved. First two linear equations for *A* are created by first assuming that player *B*

---

[2] For further detail showing that the source & target space of *T* (i.e. $S_A \times S_B$) does indeed fulfill the qualifications for a sphere in 2-space refer to *Games and Decisions* by Luce & Raffia P.393

[3] For more information on using linear programming to solve more complex games take a look at *Mathematical Introduction to Linear Programming and Game Theory* by Louis Brickman

chooses *G*, and then assuming that *B* chooses *C*. Next, the two equations are set equal to each other resulting in a value which can be used to calculate the optimal mixed strategy.

$$\begin{aligned} y_1 &= 3x + 2(1-x) \\ y_2 &= x + 4(1-x) \\ 3x + 2(1-x) &= x + 4(1-x) \\ x &= 1/2 \end{aligned}$$

The mixed nash equilibrium is (x, 1-x) = (½, ½) with expected payoff (1/2)3 + 2(1/2) = 2.5 = 1(1/2) + 4(1/2). The same is analysis is done for player *B*:

$$\begin{aligned} y_3 &= 3x + (1-x) \\ y_4 &= 2x + 4(1-x) \\ 3x + (1-x) &= 2x + 4(1-x) \\ x &= 3/4 \end{aligned}$$

The mixed nash equilibrium is (3/4, ¼) with identical expected payoff 3(3/4) + (1-3/4) = 2.5 = 2(3/4)+4(1/4). The expected payoffs for both players are equal just as expected. For m x n games (i.e. games with *m* strategies for player *A* and *n* strategies for *B*) the procedure is similar but much more complicated: many more equations will exist and instead of setting equations to each other they are held to inequality constraints.

Solving Using Numerical Estimation

The games that have thus far been analyzed have been rather simplistic compared to what a real-world game with many players and many pure strategies would look like. However, this does not mean that game theory is not practical for situations with numbers amount of players and pure strategies. The definitions and theorems within the framework of game theory can be translated into computer algorithms to solve very complex games. Below is the minimax theorem operating as JavaScript code providing the ability to automate the solving of 2-person, 2x2, zero-sum games (note that it will not properly report the nash equilibrium on games that contain more than one nash equilibrium).[4]

The results of the minimax algorithm implemented in JavaScript confirm the earlier analysis we performed on figure 6 & 7. Using a 2x2 version of the figure 6 game (with dominated strategies D & U removed) the code below properly predicted the minimax value as 5 and equilibrium pair as ((0,1),(0,1)). Although not as exact as the analysis done by solving directly, the code below estimated the minimax for figure 7 to be 2.499 with an equilibrium pair ((0.5, 0.5), (0.756, 0.244)). Gambit[5] or advanced computer algebra systems could be used to solve *m x n* games, or games with more than two players.

```
var gameMatrix = [[2,4],[6,5]];
var rowCount = gameMatrix.length;
var columnCount = gameMatrix[0].length;

// an accuracy measure; anymore than 1000 will crash modern browsers
var steps = 1000;

// mutual payoff function

function M(u, s) {
  var expectedPayoff = 0;

  for(var i = 0; i < rowCount; i++) {
    for(var j = 0; j < columnCount; j++) {
      expectedPayoff += u[i] * s[j] * gameMatrix[i][j];
    }
  }

  return expectedPayoff
}

var insideMin = 100000, outsideMax = -100000;
var nashEquilibrium = [[], []];

// minimax implementation (i.e. finding security value)

for(x = 0; x < steps + 1; x++) {
  for(y = 0; y < steps; y++) {
```

---

[4] This code can be modified and run through an online JavaScript console located at <http://jsbin.com/ofovo3/2>

[5] Gambit is a software package specifically designed to help solve game theoretic problems <http://www.gambit-project.org/doc/index.html>

```
    var prevMin = insideMin;
    var payoff = M([x/steps, 1-x/steps], [y/steps, 1-y/steps])
    insideMin = Math.min(insideMin, payoff);

    // this is for recording the nash equilibrium value
    // a new local min != a global maximum; must check both
    if(prevMin != insideMin && outsideMax != Math.max(insideMin, outsideMax)) {
      nashEquilibrium[1] = [y/steps, 1-y/steps];
    }
  }

  var prevMax = outsideMax;
  outsideMax = Math.max(insideMin, outsideMax);

  if(prevMax != outsideMax || outsideMax == insideMin) {
    nashEquilibrium[0] = [x/steps, 1-x/steps];
  }

  // reset the local minimum
  insideMin = 100000;
}

document.write('minimax: ' + outsideMax + "<br/>");
document.write('nash eq: ((' +
  nashEquilibrium[0][0] + ", " + nashEquilibrium[0][1] + "), (" +
  nashEquilibrium[1][0] + ", " + nashEquilibrium[1][1] +"))"
);
```

III. Conclusion

Game theory is at a very interesting point in its development as a field of study. Almost all of the major development in game theory has been done with the last hundred years and new material furthering the development of the field is constantly being released. It is unique in that it has found application in various disciplines such as biology, artificial intelligence implementations, and economics. Despite a somewhat widespread acceptance of game theory as a useful tool for analysis in a variety of situations some consider it to be inherently flawed “it flunks the main test of any scientific theory: The ability to make empirically testable predictions” (Mandel, 2009).

A decision maker must be aware of all the tools and resources that can be used to harness and interpret information in order to be able to make the optimal decision. Game theory should not be thought of a standalone tool used to make decisions or determine outcomes. The conclusions that the theoretical mathematics upon which game theory is based bring us to, allow us to assign certain attributes to specific strategic situations in order to aid one in understanding what is beneficial for both himself and his opponent (or ally). These attributes and characteristics which game theory allows us to ascertain from a situation enable additional facets of a given competitive situation to be revealed, making game theory an indispensible tool for a decision maker to have in his intellectual toolbox.

References:

Antoine Augustin Cournot. (2004). *Encyclopedia of World Biography*. Encyclopedia.com. Retrieved November from http://www.encyclopedia.com/doc/1G2-3404701543.html

Avinash Dixit and Barry Nalebuff. (2008). Game Theory. *The Concise Encyclopedia of Economics*. Library of Economics and Liberty. Retrieved from http://econlib.org/library/Enc/GameTheory.html

Comlay, E. (2010, November 10). Google to give staff a 10 percent pay rise: reports . *Business & Financial News, Breaking US & International News*. Retrieved from http://www.reuters.com/article/idUSTRE6A90LT20101110

Dixit, A. K., Skeath, S., & Reiley, D. (2009). *Games of Strategy* (3rd ed.). New York: W. W. Norton & Co..

Ekelund, R. B., & Hébert, R. F. (2007). *A History of Economic Theory and Method* (5th ed.). Long Grove, Ill.: Waveland Press.

Luce, R. D., & Raiffa, H. (1957). *Games and Decision: Introduction and Critical Survey*. New York: John Wiley & Sons, Inc.

Fonseca, G. L. (n.d.). John von Neumann. *New School of Economics*. Retrieved from http://homepage.newschool.edu/het/profiles/neumann.htm

Hawkins, A. (2009, June 3). Prisoner's Dilemma. *Forbes*. Retrieved November 26, 2010, from http://www.forbes.com/forbes/2009/0622/trustbusters-consumers-monopolies-heads-up.html

John von Neumann. (2008). *The Concise Encyclopedia of Economics*. Library of Economics and Liberty. Retrieved http://www.econlib.org/library/Enc/bios/Neumann.html

John F. Nash Jr. (2008). *The Concise Encyclopedia of Economics*. Library of Economics and Liberty. Retrieved from http://www.econlib.org/library/Enc/bios/Nash.html

Kuhn, H. W. (1953). Extensive Games and the Problem of Information. *Annals of Mathematical Studies*, 2, 189 - 216.

Luper, S. (n.d.). Ethical Egoism: Is Duty a Matter of Self-Enhancement?. *Trinity University*. Retrieved from http://www.trinity.edu/departments/philosophy/sluper/CHAPTER%207guidetoethics.htm

Mandel, M. (2005, September 11). A Nobel Letdown in Economics. *Business Week*. Retrieved from http://www.businessweek.com/bwdaily/dnflash/oct2005/nf20051011_3028_db084.htm

Mendelson, E. (2004). *Introducing Game Theory and Its Applications* . Boca Raton: Chapman & Hall/Crc.

Milnor, J. (1998). John Nash and “A Beautiful Mind”. *Notices*, *45*(10). Retrieved from http://www.ams.org/notices/199810/milnor.pdf

Mandel, M. (2005, September 11). A Nobel Letdown in Economics. *Business Week*. Retrieved from http://www.businessweek.com/bwdaily/dnflash/oct2005/nf20051011_3028_db084.htm

Myhrvold, N. (1999, March 21). John von Neumann. *TIME.com*. Retrieved from http://www.time.com/time/magazine/article/0,9171,21839,00.html

Stone, B., & Vance, A. (2009, May 22). Apple′s Management Obsessed With Secrecy. *Breaking News, World News & Multimedia*. Retrieved November 10, 2010, from http://www.nytimes.com/2009/06/23/technology/23apple.html?hp

Straffin, P. D. (1996). Game Theory and Strategy (New Mathematical Library). Washington: The Mathematical Association Of America